INTRODUCTION

During these last two decades, the image denoising scene has been dominated by the ROF method [2]. The method has, however, some significant drawbacks, such as the loss of image contrast, the smearing of corners, and the staircase effect. The following model was proposed by Zhu and Chan [4] to remedy these drawbacks:

\[
\mathcal{J}(u) = \epsilon \int_{\Omega} \sqrt{1 + |\nabla v|^2} \, dx + \frac{1}{2} \int |f - v|^2 \, dx,
\]

where \(v \in V\),

\[
\mathcal{J}(v) \leq \mathcal{J}(v), \forall v \in V,
\]

with

where \(\Omega\) is a suitable space of restored functions. This model is commonly known these days as the \(L^1\)-mean curvature denoising model. Our goal was to find a simpler, finite element based alternative to the solution algorithm described in [5].

AUGMENTED LAGRANGIAN FORMULATIION

Let us define

\[
T = \epsilon \left[ V \times E_{\Omega} \times H(\Omega) \right] \times \left[ (L^1(\Omega))^2 \times (L^2(\Omega))^2 \times L^2(\Omega) \right],
\]

where

\[
H(\Omega) = \left\{ q \in (L^2(\Omega))^2 : \nabla q \in L^2(\Omega) \right\},
\]

and

\[
E_{\Omega} = \left\{ (q_1, q_2) \in (L^2(\Omega))^2 : q_2 \mid_{\partial \Omega} = -q_1 \right\}.
\]

The minimization problem (1) is associated with the following augmented Lagrangian functional \(L_s\) :

\[
L_s(T) = \epsilon \int_{\Omega} |\nabla v|^2 \, dx + \frac{1}{2} \int |f - v|^2 \, dx + \frac{r_1}{2} \int |\nabla q_1 - \nabla q_2|^2 \, dx + \frac{r_2}{2} \int |q_1 - q_2|^2 \, dx
\]

where \((q_1, q_2) \in E_{\Omega} \).

SUBPROBLEMS

The resulting saddle-point problem is solved using a particular alternating direction method of multipliers. This leads to an iterative sequential solution of four subproblems.

Let \((u^*, p^1, p^2, \Sigma^1, \Sigma^2) \in T\). The first subproblem involves a pair \((p^1, p^2) \in E_{\Omega}\) and \(p^1, p^2\) can be solved point-wise from

\[
x = \arg \min_{x \in \mathbb{R}^2} \left( \frac{\rho}{\sqrt{1 + \rho^2}} \right) \left( b_1 + b_2 \sqrt{1 + \rho^2} \right) y,
\]

where \(b_1, b_2 \in \mathbb{R}^2\) depend on the other variables. Or, alternatively,

\[
x = \frac{\rho}{\sqrt{1 + \rho^2}} \left( b_1 + b_2 \sqrt{1 + \rho^2} \right) y,
\]

where

\[
\rho = \arg \min_{\rho \geq 0} \left[ \frac{\alpha}{2} \left( r_1 + r_2 \right) + \frac{1}{2} \left( - \sigma + b_1 + b_2 \sqrt{1 + \rho^2} \right) \right]
\]

The less simplified, two-dimensional form (7) is solved using the Newton's method.

The second subproblem is of the form

\[
r_2 \int_\Omega \nabla p^1 \cdot \nabla q_1 \, dx + r_3 \int_\Omega \nabla p^2 \cdot \nabla q_2 \, dx = 0,
\]

and it is solved using the conjugate gradient algorithm.

The third subproblem is of the form

\[
q^{n+1} = \arg \min_{q \in L^2(\Omega)} \frac{1}{2} \int |\nabla q|^2 \, dx - \frac{1}{2} \int \left( r_2 (p^1 + \Sigma^1) \cdot q \, dx + r_3 (p^2 + \Sigma^2) \cdot q \, dx \right)
\]

and is has a closed-form solution. Finally, fourth subproblem is of the form

\[
r_1 \int_\Omega \nabla u^{n+1} \cdot \nabla v \, dx + \frac{1}{2} \int (u^{n+1} - f) v \, dx = 0,
\]

and it is solved using the PSCR method [1, 3].

FINITE ELEMENT REALIZATION

The domain \(\Omega\) is triangulated using an uniform finite element space \(V_h = \{ v \in C^0(\Omega) : v|_{\Omega} \in P_1, \forall T \in \mathcal{T}_h \}\), where \(P_1\) is the space of the polynomials of two variables of degree \(\leq 1\).

The space \(H(\Omega)\) is approximated by the piecewise linear finite element space \(V_q = \{ q \in (L^2(\Omega))^2 : q_1|_{\partial \Omega} = -q_2|_{\partial \Omega} \}\).

The divergence operator is approximated, by using an appropriate discretization Green’s formula and the trapezoidal rule, as follows:

\[
(\div \Sigma, q)(X) = -\frac{1}{|T|} \int_{\partial T} q \cdot \nabla w_{\Sigma} \, dx,
\]

where \(X\) is a vertex that does not belong to \(\partial \Omega\), \(\Omega\) is the polygon that is the union of those triangles of \(\mathcal{T}_h\) that have \(X\) as a common vertex, \(|\Omega|\) is the measure of \(\Omega\), \(|T|\) and the shape function \(w_{\Sigma} \in V_q\) is defined as

\[
\begin{align*}
\left. w_{\Sigma}(X) \right|_{\partial \Omega} = 1, & \quad \left. w_{\Sigma}(X) \right|_{\partial T} = 0, \quad k \neq j.
\end{align*}
\]

REFERENCE


ACKNOWLEDGMENTS

The research of the first author was supported by the Academy of Finland, grant #252549.